MA 441, Fall 2012

Assignment 10

This homework is due *Tuesday* Nov 27.

There are total 19 points in this assignment. 17 points is considered 100%. If you go over 17 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 5.2, 5.3 in Bartle–Sherbert.

(1) [3pt] (5.2.6) Let f, g be defined on $\mathbb{R}$ and let $c \in \mathbb{R}$. suppose that $\lim\limits_{x\to c} f = b$ and that g is continuous at b. Show that $\lim\limits_{x\to c} g(f(x)) = g(b)$.
(*Hint:* (Re)define f to be b at c, apply composition of continuous functions.)
Note. This statement says that lim and a *continuous* function can be swapped: $\lim\limits_{x\to c} g(f(x)) = g(\lim\limits_{x\to c} f(x))$.

(2) [2pt] (5.2.7) Give an example of a function $f : [0, 1] \to \mathbb{R}$ that is discontinuous at every point of $[0, 1]$ but such that $|f|$ is continuous on $[0, 1]$.

(3) [3pt] (5.2.15) Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous at a point c, and let $h(x) = \max\{f(x), g(x)\}$ for $x \in \mathbb{R}$. Show that $h(x) = \frac{1}{2}(f(x)+g(x)+|f(x)-g(x)|)$ for all $x \in \mathbb{R}$. Use this to show that h is continuous at c.

(4) [2pt] (5.3.1) Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be a continuous function such that $f(x) > 0$ for all $x \in I$. Prove that there is a number $\alpha > 0$ such that $f(x) \geq \alpha$ for all $x \in I$.

(5) [2pt] (Part of 5.3.5) Show that the polynomial $x^4 + 7x^3 - 9$ has at least two real roots.

(6) [3pt] (5.3.6) Let f be continuous on the interval $[0, 1]$ to $\mathbb{R}$ and such that $f(0) = f(1)$. Prove that there exists a point $c \in [0, \frac{1}{2}]$ such that $f(c) = f(c + \frac{1}{2})$. (*Hint:* Consider $g(x) = f(x) - f(x + \frac{1}{2})$.)
Note. Therefore, there are, at any time, antipodal points on the earth's equator that have the same temperature.

(7) [4pt] (5.3.13) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $\lim\limits_{x\to-\infty} f = \lim\limits_{x\to\infty} f = 0$. Prove that f is bounded on $\mathbb{R}$ and attains either a maximum or a minimum on $\mathbb{R}$. Give an example to show that both a maximum and a minimum need not be attained. (*Hint:* There are several ways to approach this. One way is to (a) assume for definiteness that there is $a \in \mathbb{R}$ such that $f(a) > 0$, then (b) using limits at infinity find an interval $[-M, M]$, *outside* of which $|f(x)| < f(a)$, so (c) maximum on $[-M, M]$ is an absolute maximum on $\mathbb{R}$.)